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Affine surfaces in 4-dimensional affine space with planar geodesics

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Abstract

In this paper we study nondegenerate affine immersions, as introduced by Nomizu and Pinkall, of an affine surface (M, ∇) in the 4-dimensional affine space $(\mathbb{R}^4, D)$. Using an existence and uniqueness theorem, we classify all such immersions ϕ which map the ∇- geodesics of M into planar curves in $\mathbb{R}^4$. These theorems generalize results previously obtained in the Riemannian, the equiaffine and the centroaffine case.

Subject class: 53A15

1 Introduction

We consider the standard affine space $\mathbb{R}^{n+p}$ equipped with its standard connection D. Let M^n be a manifold equipped with a torsion free affine connection ∇ and let $\phi : (M^n, \nabla) \to (\mathbb{R}^{n+p}, D)$, $p \geq 1$ be an immersion. Following [1], we call ϕ an affine immersion if there exists a transversal p-dimensional bundle such that

$$D_X \phi_*(Y) - \phi_*(\nabla_X Y) \in \sigma, \tag{1}$$

for all vector fields X and Y which are tangent to M^n. It is immediately clear that if we equip $\mathbb{R}^{n+p}$ with a semi-Riemannian metric and take for σ the normal bundle, then nondegenerate isometric immersions provide examples of affine immersions. Also the equiaffine immersions, in the sense of Blaschke for hypersurfaces, and in the sense of [6], [7] or [3] for higher codimensions provide examples of affine immersions.

For an affine immersion it is possible to introduce a bilinear form h, called the second fundamental form, which takes values in the tranversal bundle σ by

$$h(X, Y) = D_X \phi_*(Y) - \phi_*(\nabla_X Y) \in \sigma, \tag{2}$$

Since ∇ is a torsion free affine connection, h is symmetric in X and Y. Let ξ be a vector field which takes values in σ. Similarly, as for isometric immersions, we can now introduce a normal connection $\nabla^\perp$ and Weingarten operators A by decomposing $D_X \xi$ into a tangential part and a part in the direction of σ, i.e. we have the Weingarten formula which states that

$$D_X \xi = -\phi_*(A_\xi X) + \nabla^\perp_X \xi. \tag{3}$$

In case there is no confusion possible we will always identify M with its image in $\mathbb{R}^{n+p}$.

We now restrict to the special case of surfaces in $\mathbb{R}^4$. In case that ξ_1 and ξ_2 form a local basis of the transversal plane σ it is convenient to write:

$$h(X, Y) = h^1(X, Y)\xi_1 + h^2(X, Y)\xi_2.$$

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In Section 2 we recall some basic facts for affine surfaces in $\mathbb{R}^4$. In particular, we introduce the notion of nondegenerate affine surfaces by inducing a conformal structure on M. Next we assume that the images of all geodesic of the connection ∇ are planar curves in $\mathbb{R}^4$. We will show that this naturally induces a 1-form α and a symmetric bilinear form $\tilde{g}$ on our surface.

After that we deal with the positive definite case and the indefinite case separately in Section 3 and 4. In both cases, we choose a special frame, which depends on the properties of this 1-form and the symmetric bilinear form $\tilde{g}$. By a case by case analysis of the equations of Gauss, Codazzi and Ricci we then conclude each part with an existence or a non-existence result. Surfaces with planar geodesics in a 4-dimensional affine space have been previously in [4] and [5] for respectively the centroaffine transveral plane bundle and for the Burstin-Mayer, the Weise-Klingenberg and the equiaffine tranversal plane bundle. Here in this paper, we will not make any assumptions about the transversal plane bundle, thus generalizing the previous results. In an appendix we describe a method which can be used to obtain a more explicit description of the examples we obtained by applying the standard existence and uniqueness theorem. Putting the results in the different sections together then yields a classification.

2 Preliminaries

From now on, we will always assume that $\phi : (M^2, \nabla) \to (R^4, D)$ is an affine immersion. For convenience, we fix a volumeform Ω on $\mathbb{R}^4$ given by the determinant. Then, we recall from [3] that a conformal structure can be introduced in the following way. Let $u = \{X_1, X_2\}$ be a local differentiable frame on a neighbourhood U of a point p. Then we use the frame u to define a symmetric bilinear form on M by

$$G_u(Y,Z) = \frac{1}{2}([X_1, X_2, D_Y X_1, D_Z X_2] + [X_1, X_2, D_Z X_1, D_Y X_2]).$$

Of course, the symmetric bilinear form G_u depends on the choice of frame u. Assume that $u = \{X_1, X_2\}$ and $v = \{Y_1, Y_2\}$ are two local frames on a neighborhood U of a point p of M. Thus there exist functions a, b, c and d with $ad - bc \neq 0$, defined on U, such that

$$Y_1 = aX_1 + bX_2, \tag{4}$$
$$Y_2 = cX_1 + dX_2. \tag{5}$$

Then the symmetric bilinear forms G_u and G_v are related by

$$G_v(Y,Z) = (ad - bc)^2 G_u(Y,Z).$$

Which implies that the nondegeneracy and the signature of G_u are independent of the choice of frame u. Therefore, we call M nondegenerate if G_u is nondegenerate. From now on, we will always assume that M is nondegenerate. In case that G_u is definite, we call M definite, whereas otherwise we call M indefinite. We will then use the following choice of basis:

Lemma 1 *Let M be definite. Then there exists a local basis X_1 and X_2 of the tangent space and a local basis ξ_1 and ξ_2 of the transversal space σ such that*

$$h(X_1, X_1) = \xi_1,$$
$$h(X_1, X_2) = \xi_2,$$
$$h(X_2, X_2) = -\xi_1.$$

Moreover any 2 such basis $\{X_1, X_2, \xi_1, \xi_2\}$ and $\{Y_1, Y_2, \eta_1, \eta_2\}$ are related by

$$Y_1 = r(\cos\theta X_1 + \sin\theta X_2),$$
$$Y_2 = \pm r(-\sin\theta X_1 + \cos\theta X_2),$$
$$\eta_1 = r^2(\cos 2\theta \xi_1 + \sin 2\theta \xi_2),$$
$$\eta_2 = \pm r(-\sin 2\theta \xi_1 + \cos 2\theta \xi_2),$$

where r and θ are arbitrary local functions.

Lemma 2 *Let M be indefinite. Then there exists a local basis X_1 and X_2 of the tangent space and a local basis ξ_1 and ξ_2 of the transversal space σ such that*

$$\begin{aligned} h(X_1, X_1) &= \xi_1, \\ h(X_1, X_2) &= 0, \\ h(X_2, X_2) &= \xi_2. \end{aligned}$$

Moreover any 2 such basis $\{X_1, X_2, \xi_1, \xi_2\}$ and $\{Y_1, Y_2, \eta_1, \eta_2\}$ are related by

$$\begin{aligned} Y_1 &= rX_1, \\ Y_2 &= sX_2, \\ \eta_1 &= r^2\xi_1, \\ \eta_2 &= s^2\xi_2, \end{aligned}$$

where r and s are arbitrary local functions.

Let X_1, X_2, ξ_1, ξ_2 be the basis constructed in the previous lemmas. Then, we introduce shape operators S_1 and S_2 by

$$\begin{aligned} D_X\xi_1 &= -S_1X + \nabla^{\perp}_X\xi_1, \\ D_X\xi_2 &= -S_2X + \nabla^{\perp}_X\xi_2. \end{aligned}$$

The surface M is called

(i) affine minimal if $\operatorname{trace} S_1 = \operatorname{trace} S_2 = 0$,

(ii) proper affine umbilical if S_1 and S_2 are multiples of the identity and M is not affine minimal,

(iii) improper affine umbilical if S_1 and S_2 both vanish.

We now recall that the basic equations of Gauss, Codazzi and Ricci state respectively that

$$\begin{aligned} &R(X,Y)Z = A_{h(Y,Z)}X - A_{h(X,Z)}Y, \\ &(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z), \\ &(\nabla_X A)_\xi Y = (\nabla_Y A)_\xi X, \\ &R^{\perp}(X,Y)\xi = h(X, A_\xi Y) - h(Y, A_\xi X). \end{aligned}$$

The cubic form C is defined by $C(X,Y,Z) = (\nabla_X h)(Y,Z)$ and the above equations state that the cubic form is symmetric in X, Y and Z. It is shown in [3] that the vanishing of C characterizes complex curves and products of planar curves.

We now assume that the image of every geodesic is always a planar curve. This is characterized in the following lemma:

Lemma 3 *Let $\phi : (M^2, \nabla) \to \mathbb{R}^4$ is a nondegenerate affine immersion such that the images of the geodesics are planar curves in $\mathbb{R}^4$. Then there exists a 1-form α and a symmetric bilinear form $\tilde{g}$ such that*

$$\begin{aligned} A_{h(v,v)}v &= \tilde{g}(v,v)v \\ (\nabla_v h)(v,v) &= \alpha(v)h(v,v), \end{aligned}$$

for any vector v.

Proof: Let v be an arbitrary non-zero vector . Denote by γ the image in $\mathbb{R}^4$ of the ∇-geodesic determined by v. Then, we have

$$\begin{aligned} \gamma''(v) &= h(v,v) \\ \gamma'''(v) &= -A_{h(v,v)}v + (\nabla_v h)(v,v). \end{aligned}$$

Since it follows immediately from the previous lemmas that $h(v,v)$ vanishes only for the null vector it follows that γ is a planar curve if and only if

(i) $A_{h(v,v)}v$ is proportional to v,

(ii) $(\nabla_v h)(v,v)$ is proportional with $h(v,v)$.

It follows immediately from the second equation that we can introduce a differentiable function $\alpha : TM \to \mathbb{R}$ such that $(\nabla_v h)(v,v) = \alpha(v)h(v,v)$ and such that $\alpha(rv) = r\alpha(v)$ for any number r and for any vector v.

In order to show that α is a 1-form, we take fixed vectors v and u and variables t and s. We consider

$$v(t,s) = tv + su,$$

and write $C_{xyz} = C(x,y,z)$, where x, y, z are vectors. Then it follows that

$$t^3 C_{vvv} + 3t^2 s C_{vvu} + 3ts^2 C_{uuv} + s^3 C_{uuu} = \alpha(v(t,s))(t^2 h(v,v) + 2sth(v,v) + s^2 h(u,u).$$

Looking now at a component for which $h(v,v)$ does not vanish, it follows that we can write

$$\begin{aligned} \alpha(v(t,s)) &= tf_1 + sf_2, \\ \alpha(v(0,s)) &= \alpha(su) = s\alpha(u) = sf_2, \\ \alpha(v(t,0)) &= \alpha(tv) = t\alpha(v) = tf_1. \end{aligned}$$

Consequently $\alpha(tv + su) = t\alpha(v) + s\alpha(u)$.

To obtain the symmetric bilinear form, we first write

$$A_{h(v,v)}v = \beta(v)v$$

for some function β and define

$$\tilde{g}(u,v) = \beta(u+v) - \beta(u) - \beta(v).$$

It then follows as before that $\tilde{g}$ is bilinear. ∎

Since $(\nabla_X h)(Y,Z))$ is totally symmetric, linearisation of the second equation of the previous lemma yields that

$$(\nabla_X h)(Y,Z) = \alpha(X)h(Y,Z) + \alpha(Y)h(X,Z) + \alpha(Z)h(X,Y). \tag{6}$$

For submanifolds satisfying the above condition, it is said that C is divisible by h. Affine hypersurfaces satisfying this condition were studied in [2].

3 Nondegenerate definite surfaces

In this section, we will assume that M^2 is a definite surface in $\mathbb{R}^4$. Then, we have the following lemma.

Lemma 4 *Let X_1, X_2, ξ_1, ξ_2 be a basis constructed as in the previous section. Then, $A_{h(v,v)}v$ is a multiple of v for every vector v if and only if*

$$\begin{aligned} S_1X_1 &= \lambda_1 X_1, \\ S_1X_2 &= \lambda_2 X_2, \\ S_2X_1 &= \mu X_1 + \tfrac{1}{2}(\lambda_1 - \lambda_2)X_2, \\ S_2X_2 &= \tfrac{1}{2}(\lambda_1 - \lambda_2)X_1 + \mu X_2. \end{aligned}$$

Proof: Since $\xi_1 = h(X_1,X_1) = -h(X_2,X_2)$, it follows immediately that we can write

$$\begin{aligned} S_1X_1 &= \lambda_1 X_1, \\ S_1X_2 &= \lambda_2 X_2. \end{aligned}$$

We now write

$$S_2 X_1 = a_{11} X_1 + a_{12} X_2,$$
$$S_2 X_2 = a_{21} X_1 + a_{22} X_2,$$

and we take $v = aX_1 + bX_2$. Then it follows that

$$\begin{aligned} A_{h(v,v)} v &= A_{(a^2-b^2)\xi_1 + 2ab\xi_2}(aX_1 + bX_2) \\ &= a(a^2 - b^2)\lambda_1 X_1 + b(a^2 - b^2)\lambda_2 X_2 + 2a^2 b(a_{11}X_1 + a_{12}X_2) + 2ab^2(a_{21}X_1 + a_{22}X_2) \\ &= (a(a^2 - b^2)\lambda_1 + 2a^2 b a_{11} + 2ab^2 a_{21})X_1 + (b(a^2 - b^2)\lambda_2 + 2a^2 b a_{12} + 2ab^2 a_{22})X_2. \end{aligned}$$

It follows that this is a multiple of the vector v if and only if

$$ab(a^2 - b^2)\lambda_1 + 2a^2b^2 a_{11} + 2ab^3 a_{21} - ab(a^2 - b^2)\lambda_2 - 2a^3 b a_{12} - 2a^2 b^2 a_{22} = 0.$$

Since the above equation has to be valued for all values of a and b it follows that

$$a_{11} = a_{22} = \mu,$$
$$a_{12} = a_{21} = \tfrac{1}{2}(\lambda_1 - \lambda_2).$$

∎

Using the notation of the previous lemma, it follows also that

$$\begin{aligned} A_{h(v,v)} v &= (a^3\lambda_1 + 2a^2 b\mu - ab^2\lambda_2)X_1 + (-b^3\lambda_2 + 2ab^2\lambda_2 + a^2 b\lambda_1)X_2 \\ &= (a^2\lambda_1 + 2ab\mu - b^2\lambda_2^2)v. \end{aligned}$$

This implies that the symmetric bilinear form $\tilde{g}$ introduced in the previous section is determined by

$$\tilde{g}(X_1, X_1) = \lambda_1,$$
$$\tilde{g}(X_1, X_2) = \mu_1,$$
$$\tilde{g}(X_2, X_2) = -\lambda_2.$$

Since the conformal structure is definite, there always exists a basis X_1 and X_2 such that μ_1 vanishes. We now consider the following cases:

(i) If M is affine minimal, $\tilde{g}$ determines the same conformal structure. In this case we choose the function r such that $\Omega(X_1, X_2, \xi_1, \xi_2) = 1$. We still have however a rotational freedom θ.

(ii) If M is proper affine umbilical, we have that $\lambda_1 = \lambda_2$. In this case we determine θ by the assumption that $\tilde{g}(X_1, X_2) = 0$ and we fix r by the assumption that $\Omega(X_1, X_2, \xi_1, \xi_2) = 1$.

(iii) If M is improper affine umbilical, we choose r such that $\Omega(X_1, X_2, \xi_1, \xi_2) = 1$ and we keep the rotational freedom

(iv) If M is none of the above, we determine θ by the assumption that $\tilde{g}(X_1, X_2) = 0$ and we fix r by the assumption that $\Omega(X_1, X_2, \xi_1, \xi_2) = 1$.

We now take the basis constructed before and introduce local functions by

$$\nabla_{X_1} X_1 = a_1 X_1 + a_2 X_2,$$
$$\nabla_{X_1} X_2 = a_3 X_1 + a_4 X_2,$$
$$\nabla_{X_1} X_2 = a_5 X_1 + a_6 X_2,$$
$$\nabla_{X_2} X_2 = a_7 X_1 + a_8 X_2,$$
$$\nabla^{\perp}_{X_1} \xi_1 = d_1 \xi_1 + d_2 \xi_2,$$
$$\nabla^{\perp}_{X_1} \xi_2 = d_3 \xi_1 + d_4 \xi_2,$$
$$\nabla^{\perp}_{X_2} \xi_1 = d_5 \xi_1 + d_6 \xi_2,$$
$$\nabla^{\perp}_{X_2} \xi_2 = d_7 \xi_1 + d_8 \xi_2.$$

Since in all cases $\Omega(X_1, X_2, \xi_1, \xi_2) = 1$, we have that

$$a_1 + a_4 + d_1 + d_4 = 0 = a_5 + a_8 + d_5 + d_8 = 0.$$

Computing the Gauss, Codazzi and Ricci equations, together with the condition that C is divisible by h it then follows by a long but straightforward computation, which will be illustrated by an example in the indefinite case in the next section, that

Lemma 5 *Let $\phi : (M^2, \nabla) \to \mathbb{R}^4$ be an improper definite affine sphere whose geodesics are planar curves in $\mathbb{R}^4$. Then we have that*

$$\begin{array}{llll} a_1 = 7a_4, & a_3 = 6a_5 - a_2, & a_6 = 6a_4 - a_7, & a_8 = 7a_5, \\ d_1 = -4a_4, & d_2 = 2a_2, & d_3 = -2a_2, & d_4 = -4a_4, \\ d_5 = -4a_5, & d_6 = -2a_7, & d_7 = 2a_7, & d_8 = -4a_5, \end{array}$$

where the functions a_2, a_3, a_4 and a_5 satisfy the following system of differential equations:

$$\begin{aligned} X_1(a_4) &= a_4^2 + a_2 a_5, \\ X_2(a_4) &= a_5(a_4 - a_7), \\ X_1(a_5) &= a_4(a_5 - a_2), \\ X_2(a_5) &= a_5^2 + a_4 a_7, \\ X_2(a_2) + X_1(a_7) - a_2^2 - a_7^2 + 5a_4 a_7 + 5a_2 a_5 &= 0. \end{aligned}$$

Theorem 1 *There does not exist a definite affine minimal immersion $\phi : (M^2, \nabla) \to \mathbb{R}^4$ whose geodesics are planar curves in $\mathbb{R}^4$ and which is not an improper affine sphere.*

Lemma 6 *Let $\phi : (M^2, \nabla) \to \mathbb{R}^4$ be a proper definite umbilical immersion whose geodesics are planar curves in $\mathbb{R}^4$. Then we have that*

$$\begin{array}{llll} a_1 = 7a_4, & a_2 = 0, & a_3 = 6a_5, & \\ a_6 = 6a_4, & a_7 = 0, & a_8 = 7a_5, & \\ d_1 = -4a_4, & d_2 = 0, & d_3 = 0, & d_4 = -4a_4, \\ d_5 = -4a_5, & d_6 = 0, & d_7 = 0, & d_8 = -4a_5, \end{array}$$

where the functions a_4, a_5 and λ_1 satisfy the following system of differential equations:

$$\begin{aligned} X_1(a_4) &= a_4^2 - \tfrac{1}{6}\lambda_1, \\ X_2(a_4) &= a_5 a_4, \\ X_1(a_5) &= a_4 a_5, \\ X_2(a_5) &= a_5^2 + \tfrac{1}{6}\lambda_1, \\ X_1(\lambda_1) &= -4a_4\lambda_1, \\ X_2(\lambda_1) &= -4a_5\lambda_1. \end{aligned}$$

Lemma 7 *Let $\phi : (M^2, \nabla) \to \mathbb{R}^4$ be a definite immersion whose geodesics are planar curves in $\mathbb{R}^4$, which at no point is affine minimal or affine umbilical. Then we have that*

$$\begin{array}{llll} a_1 = 7a_4, & a_2 = 3a_5\frac{\lambda_1-\lambda_2}{\lambda_1+\lambda_2}, & a_3 = 3a_5\frac{\lambda_1+3\lambda_2}{\lambda_1+\lambda_2}, & \\ a_6 = 3a_4\frac{3\lambda_1+\lambda_2}{\lambda_1+\lambda_2}, & a_7 = -3a_4\frac{\lambda_1-\lambda_2}{\lambda_1+\lambda_2}, & a_8 = 7a_5, & \\ d_1 = -4a_4, & d_2 = 6a_5\frac{\lambda_1-\lambda_2}{\lambda_1+\lambda_2}, & d_3 = -6a_5\frac{\lambda_1-\lambda_2}{\lambda_1+\lambda_2}, & d_4 = -4a_4, \\ d_5 = -4a_5, & d_6 = 6a_4\frac{\lambda_1-\lambda_2}{\lambda_1+\lambda_2}, & d_7 = -6a_4\frac{\lambda_1-\lambda_2}{\lambda_1+\lambda_2}, & d_8 = -4a_5, \end{array}$$

where the functions a_4, a_5, λ_1 and λ_2 satisfy the following system of differential equations:

$$\begin{aligned}
X_1(a_4) &= a_4^2 + \tfrac{1}{6}(\lambda_1 - 2\lambda_2) + 3a_5^2\tfrac{\lambda_1-\lambda_2}{\lambda_1+\lambda_2},\\
X_2(a_4) &= a_5 a_4 \tfrac{4\lambda_1-2\lambda_2}{\lambda_1+\lambda_2},\\
X_1(a_5) &= a_4 a_5 \tfrac{-2\lambda_1+4\lambda_2}{\lambda_1+\lambda_2},\\
X_2(a_5) &= a_5^2 + \tfrac{1}{6}(2\lambda_1 - \lambda_2) - 3a_4^2\tfrac{\lambda_1-\lambda_2}{\lambda_1+\lambda_2},\\
X_1(\lambda_1) &= -4a_4\lambda_1,\\
X_2(\lambda_1) &= a_5(-10\lambda_1 + 6\lambda_2),\\
X_1(\lambda_2) &= a_4(6\lambda_1 - 10\lambda_2),\\
X_2(\lambda_2) &= -4a_5\lambda_2.
\end{aligned}$$

Next we investigate what happens in each of the above cases further. First, we have the following.

Lemma 8 *Let $\phi : (M^2, \nabla) \to \mathbb{R}^4$ be an improper definite affine sphere whose geodesics are planar curves in $\mathbb{R}^4$. Then there exist an angle θ such that with respect to the new frame we have that $a_2 = a_7 = 0$.*

Proof: We take

$$\begin{aligned}
Y_1 &= \cos\theta X_1 + \sin\theta X_2,\\
Y_2 &= -\sin\theta X_1 + \cos\theta X_2.
\end{aligned}$$

Then, it follows by a straightforward computation that

$$\begin{aligned}
\nabla_{Y_1}Y_1 &= (Y_1(\theta) + a_2\cos\theta - a_7\sin\theta)Y_2 && (7)\\
&\quad + 7(\cos\theta a_4 + \sin\theta a_5)Y_1, && (8)\\
\nabla_{Y_2}Y_2 &= (-Y_2(\theta) + \sin\theta a_2 + \cos\theta a_7)Y_1 && (9)\\
&\quad + 7(-\sin\theta a_4 + \cos\theta a_5). && (10)
\end{aligned}$$

Hence the existence of the desired frame is equivalent with the existence of a function θ satisfying

$$\begin{aligned}
Y_1(\theta) &= a_2\cos\theta - a_7\sin\theta,\\
Y_2(\theta) &= -a_2\sin\theta - a_7\cos\theta,
\end{aligned}$$

or equivalently

$$\begin{aligned}
X_1(\theta) &= a_2, && (11)\\
X_2(\theta) &= -a_7. && (12)
\end{aligned}$$

The integrability condition of the system of differential equations given by (11) and (12) then states that

$$\begin{aligned}
X_2(a_2) + X_1(a_7) &= (X_2X_1 - X_1X_2)\theta\\
&= [X_2, X_1]\theta\\
&= -(a_3 - a_5)X_1(\theta) - (a_4 - a_6)X_2(\theta)\\
&= -(5a_5 - a_2)a_2 - (5a_4 - a_7)a_7,
\end{aligned}$$

which is satisfied. ∎

Theorem 2 *Let $\phi : (M^2, \nabla) \to \mathbb{R}^4$ be an improper definite affine sphere whose geodesics are planar curves in $\mathbb{R}^4$. Then, by restricting to an open dense subset of M, each point has a neighborhood which is affine congruent to either*

(i) the complex paraboloid $(u, v, \frac{1}{2}(u^2 - v^2), uv)$,

(ii) the surface $(v^{-6}, v^{-6}u, \frac{1}{2}u^2v^{-6} - \frac{1}{72}v^6, \frac{1}{6}u)$

(iii) the surface $\frac{1}{(-6c^5u-6cv)}(\frac{1}{2}(c^{10}u^2 - v^2), c^5uv, u, 1)$.

Proof: We take the frame constructed in the previous lemma. First, we assume that $a_4 = a_5 = 0$ in a neighborhood of the point p. In that case there exist coordinates u and v such that $X_1 = \frac{\partial}{\partial u}$ and $X_2 = \frac{\partial}{\partial v}$. The immersion is then determined by

$$\phi_{uu} = \xi_1 = -\phi_{vv},$$
$$\phi_{uv} = \xi_2,$$

where both ξ_1 and ξ_2 are constant vector fields. Integrating the above equations then yields the complex paraboloid.

Next we assume that $a_4 = 0$ and $a_5 \neq 0$ in a neighborhood of the point that we are considering. In that case, we have that

$$[X_1, X_2] = 5a_5X_1,$$

where the function a_5 satisfies

$$X_1(a_5) = 0$$
$$X_2(a_5) = a_5^2.$$

From this we see that

$$[a_5^5X_1, X_2] = 5a_5^6X_1 - 5a_5^6X_1 = 0.$$

Hence there exist coordinates u and v such that $a_5^5X_1 = \frac{\partial}{\partial u}$ and $X_2 = \frac{\partial}{\partial v}$. The function a_5 is then a function depending only on the variable v and determined by

$$\frac{\partial a_5}{\partial v} = a_5^2. \tag{13}$$

Hence, solving (13) and applying a translation of the v coordinate, if necessary, we may assume that $a_5 = -\frac{1}{v}$. The immersion ϕ is then determined by the following system of differential equations.

$$\phi_{uu} = v^{-10}\xi_1 \tag{14}$$
$$\phi_{uv} = -6v^{-1}\phi_u + v^{-5}\xi_2 \tag{15}$$
$$\phi_{vv} = -7v^{-1}\phi_v - \xi_1 \tag{16}$$
$$\xi_{1u} = 0 \tag{17}$$
$$\xi_{2u} = 0 \tag{18}$$
$$\xi_{1v} = \frac{4}{v}\xi_1 \tag{19}$$
$$\xi_{2v} = \frac{4}{v}\xi_2 \tag{20}$$

Solving first (17), (18), (19) and (20) it follows that there exist constant vectors A_1 and A_2 such that $\xi_1 = A_1v^4$ and $\xi_2 = A_2v^4$. Substituting this into (14) if follows that we can write

$$\phi = \frac{1}{2}u^2v^{-6}A_1 + C_3(v)u + C_4(v), \tag{21}$$

where C_3 and C_4 are vector valued functions depending only on the variable v. Substituting (21) into (15) it then follows that

$$C_3'(v) = -6v^{-1}C_3(v) + A_2v^{-1}. \tag{22}$$

This implies that there exist a constant vector A_3 such that $C_3(v) = A_3v^{-6} + \frac{1}{6}A_2$. Finally it then follows from (16) that

$$C_4''(v) + 7v^{-1}C_4'(v) = -A_1v^4.$$

Therefore after a translation we have that

$$C_4(v) = -\tfrac{1}{72}A_1 v^6 + A_4 v^{-6}, \tag{23}$$

and ϕ is congruent to an open part of the surface

$$(v^{-6}, v^{-6}u, \tfrac{1}{2}u^2v^{-6} - \tfrac{1}{72}v^6, \tfrac{1}{6}u) \tag{24}$$

Finally we deal with the case that $a_4 \neq 0 \neq a_5$ in a neighborhood of the point p. In that case, we have

$$[X_1, X_2] = 5a_5X_1 - 5a_4X_2.$$

From this we see that

$$[a_5^5X_1, a_4^5X_2] = a_4^5a_5^5(5a_5X_1 - 5a_4X_2) + 5a_4^6a_5^5X_2 - 5a_4^5a_5^6X_1 = 0.$$

Hence there exist coordinates u and v such that $a_5^5X_1 = \frac{\partial}{\partial u}$ and $a_4^5X_2 = \frac{\partial}{\partial v}$. The functions a_4 and a_5 are then determined by the following system of differential equations:

$$\frac{\partial a_4}{\partial u} = a_5^5a_4^2, \tag{25}$$
$$\frac{\partial a_4}{\partial v} = a_5a_4^6, \tag{26}$$
$$\frac{\partial a_5}{\partial u} = a_5^6a_4, \tag{27}$$
$$\frac{\partial a_5}{\partial v} = a_5^2a_4^5. \tag{28}$$

From these equations it immediately follows that

$$\frac{\partial}{\partial u}\frac{a_4}{a_5} = 0, \tag{29}$$
$$\frac{\partial}{\partial v}\frac{a_4}{a_5} = 0. \tag{30}$$

Hence there exists a non-zero constant c such that $a_5 = ca_4$. The equations (25), (26), (27) and (28) then reduce to

$$\frac{\partial a_4}{\partial u} = c^5a_4^7, \tag{31}$$
$$\frac{\partial a_4}{\partial v} = ca_4^7, \tag{32}$$

Solving the above system of differential equations it follows that if necessary after a translation of the coordinates, we have that

$$a_4 = (-6c^5u - 6cv)^{-\frac{1}{6}}.$$

It then follows that the immersion ϕ is determined by the following system of differential equations:

$$\phi_{uu} = 12c^5a_4^6\phi_u + c^10a_4^{10}\xi_1, \tag{33}$$
$$\phi_{uv} = 6ca_4^6\phi_u + 6c^5a_4^6\phi_v + c^5a_4^{10}\xi_2, \tag{34}$$
$$\phi_{vv} = 12ca_4^6\phi_v - a_4^{10}\xi_1, \tag{35}$$
$$\xi_{1u} = -4c^5a_4^6\xi_1, \tag{36}$$
$$\xi_{2u} = -4c^5a_4^6\xi_2, \tag{37}$$
$$\xi_{1v} = -4ca_4^6\xi_1, \tag{38}$$
$$\xi_{2v} = -4ca_4^6\xi_2, \tag{39}$$

Solving first (36), (37), (38) and (39) it follows that there exist constant vectors A_1 and A_2 such that $a_4^4\xi_1 = A_1$ and $a_4^4\xi_2 = A_2$. Hence the system of differential equations reduces to

$$\phi_{uu} = 12c^5a_4^6\phi_u + c^{10}a_4^6A_1, \tag{40}$$
$$\phi_{uv} = 6ca_4^6\phi_u + 6c^5a_4^6\phi_v + c^5a_4^6A_2, \tag{41}$$
$$\phi_{vv} = 12ca_4^6\phi_v - a_4^6A_1, \tag{42}$$

which, since $a_4^{-6} = -6c^5u - 6cv$, can still be rewritten as

$$[(-6c^5u - 6cv)\phi]_{uu} = c^{10}A_1,$$
$$[(-6c^5u - 6cv)\phi]_{uv} = c^5 A_2,$$
$$[(-6c^5u - 6cv)\phi]_{vv} = -A_1.$$

Hence it follows that there exist constant vectors A_3, A_4 and A_5 such that

$$(-6c^5u - 6cv)\phi = (\tfrac{c^{10}u^2}{2} - \tfrac{v^2}{2})A_1 + c^5uvA_2 + A_3u + A_4v + A_5.$$

Thus by an affine transformation, together with a translation ϕ is congruent with

$$\phi(u,v) = \tfrac{1}{(-6c^5u-6cv)}(\tfrac{1}{2}(c^{10}u^2 - v^2), c^5uv, u, 1).$$

∎

Theorem 3 *Let $\phi : (M^2, \nabla) \to \mathbb{R}^4$ be a proper definite umbilical immersion whose geodesics are planar curves in $\mathbb{R}^4$. Then M is affine congruent with the immersion given by*

$$\phi(u,v) = \tfrac{1}{(\frac{1}{2}(u^2-v^2)+c)}(1, u, v, uv).$$

Proof: Since M is proper umbilical, we have that $\lambda_1 \neq 0$. As before, we have that

$$[X_1, X_2] = 5a_5X_1 - 5a_4X_2,$$

from which it follows that

$$[\lambda_1^{-\frac{5}{4}}X_1, \lambda_1^{-\frac{5}{4}}X_2] = \lambda_1^{-\frac{5}{2}}(5a_5X_1 - 5a_4X_2) + \lambda_1^{-\frac{5}{2}}5a_4X_2 - \lambda_1^{-\frac{5}{2}}5a_5X_1 = 0.$$

Hence there exist coordinates u and v such that $\lambda_1^{-\frac{5}{4}}X_1 = \frac{\partial}{\partial u}$ and $\lambda_1^{-\frac{5}{4}}X_2 = \frac{\partial}{\partial v}$. The functions a_4, a_5 and λ_1 are then determined by the following system of differential equations:

$$\tfrac{\partial a_4}{\partial u} = (a_4^2 - \tfrac{1}{6}\lambda_1)\lambda_1^{-\frac{5}{4}}, \tag{43}$$
$$\tfrac{\partial a_4}{\partial v} = a_5a_4\lambda_1^{-\frac{5}{4}}, \tag{44}$$
$$\tfrac{\partial a_5}{\partial u} = a_5a_4\lambda_1^{-\frac{5}{4}}, \tag{45}$$
$$\tfrac{\partial a_5}{\partial v} = (a_5^2 + \tfrac{1}{6}\lambda_1)\lambda_1^{-\frac{5}{4}}, \tag{46}$$
$$\tfrac{\partial \lambda_1}{\partial u} = -4a_4\lambda_1^{-\frac{1}{4}}, \tag{47}$$
$$\tfrac{\partial \lambda_1}{\partial v} = -4a_5\lambda_1^{-\frac{1}{4}}. \tag{48}$$

From these equations it immediately follows that

$$\tfrac{\partial}{\partial u}\lambda_1^{\frac{1}{4}}a_5 = 0, \tag{49}$$
$$\tfrac{\partial}{\partial v}\lambda_1^{\frac{1}{4}}a_5 = \tfrac{1}{6}, \tag{50}$$
$$\tfrac{\partial}{\partial u}\lambda_1^{\frac{1}{4}}a_4 = -\tfrac{1}{6}, \tag{51}$$
$$\tfrac{\partial}{\partial v}\lambda_1^{\frac{1}{4}}a_4 = 0. \tag{52}$$

Hence by applying a translation of the u and v coordinates we may assume that

$$\begin{aligned} \tfrac{1}{6}v &= \lambda_1^{\frac{1}{4}} a_5, \\ -\tfrac{1}{6}u &= \lambda_1^{\frac{1}{4}} a_4, \end{aligned}$$

or equivalently

$$\begin{aligned} a_4 &= -\tfrac{1}{6}u\lambda_1^{-\frac{1}{4}}, \\ a_5 &= \tfrac{1}{6}v\lambda_1^{-\frac{1}{4}}. \end{aligned}$$

Consequently λ_1 is determined as the solution of the differential equation:

$$\begin{aligned} \tfrac{\partial \lambda_1}{\partial u} &= \tfrac{2}{3}u\lambda_1^{-\frac{1}{2}}, \\ \tfrac{\partial \lambda_1}{\partial v} &= -\tfrac{2}{3}v\lambda_1^{-\frac{1}{2}}. \end{aligned}$$

Hence

$$\lambda_1^{\frac{3}{2}} = \tfrac{1}{2}(u^2 - v^2) + c.$$

It then follows that the immersion ϕ is determined by the following system of differential equations:

$$\phi_{uu} = -2u\lambda_1^{-\frac{3}{2}}\phi_u + \lambda_1^{-\frac{5}{2}}\xi_1, \tag{53}$$

$$\phi_{uv} = \lambda_1^{-\frac{3}{2}}(v\phi_u - u\phi_v) + \lambda_1^{-\frac{5}{2}}\xi_2, \tag{54}$$

$$\phi_{vv} = 2v\lambda_1^{-\frac{3}{2}}\phi_v - \lambda_1^{-\frac{5}{2}}\xi_1, \tag{55}$$

$$\xi_{1u} = -\lambda_1\phi_u + \tfrac{2}{3}u\lambda_1^{-\frac{3}{2}}\xi_1, \tag{56}$$

$$\xi_{2u} = \tfrac{2}{3}u\lambda_1^{-\frac{3}{2}}\xi_2, \tag{57}$$

$$\xi_{1v} = -\lambda_1\phi_v - \tfrac{2}{3}v\lambda_1^{-\frac{3}{2}}\xi_1, \tag{58}$$

$$\xi_{2v} = -\tfrac{2}{3}v\lambda_1^{-\frac{3}{2}}\xi_2. \tag{59}$$

Solving first (36), (37), (38) and (39) it follows that there exist constant vectors A_1 and A_2 such that $\lambda_1^{-1}\xi_1 + \phi = A_1$ and $\lambda_1^{-1}\xi_2 = A_2$. Hence the system of differential equations reduces to

$$\begin{aligned} [\lambda_1^{\frac{3}{2}}\phi]_{uu} &= A_1, \\ [\lambda_1^{\frac{3}{2}}\phi]_{uv} &= A_2, \\ [\lambda_1^{\frac{3}{2}}\phi]_{vv} &= -A_1. \end{aligned}$$

Hence it follows that there exist constant vectors A_3, A_4 and A_5 such that

$$(\tfrac{1}{2}(u^2 - v^2) + c)\phi = (\tfrac{u^2}{2} - \tfrac{v^2}{2})A_1 + uvA_2 + A_3u + A_4v + A_5.$$

Thus by an affine transformation, together with a translation ϕ is congruent with

$$\phi(u,v) = \frac{1}{(\frac{1}{2}(u^2-v^2)+c)}(1, u, v, uv).$$

∎

Theorem 4 *Consider $D \subset \mathbb{R}^2$ and let $a_4, a_5, \lambda_1, \lambda_2$ be a solution of*

$$\begin{aligned}
\tfrac{\partial}{\partial u}(a_4) &= \rho(a_4^2 + \tfrac{1}{6}(\lambda_1 - 2\lambda_2) + 3a_5^2\tfrac{\lambda_1-\lambda_2}{\lambda_1+\lambda_2}),\\
\tfrac{\partial}{\partial v}(a_4) &= \rho a_5 a_4 \tfrac{4\lambda_1-2\lambda_2}{\lambda_1+\lambda_2},\\
\tfrac{\partial}{\partial u}(a_5) &= \rho a_4 a_5 \tfrac{-2\lambda_1+4\lambda_2}{\lambda_1+\lambda_2},\\
\tfrac{\partial}{\partial v}(a_5) &= \rho(a_5^2 + \tfrac{1}{6}(2\lambda_1 - \lambda_2) - 3a_4^2\tfrac{\lambda_1-\lambda_2}{\lambda_1+\lambda_2}),\\
\tfrac{\partial}{\partial u}(\lambda_1) &= -4a_4\rho\lambda_1,\\
\tfrac{\partial}{\partial v}(\lambda_1) &= a_5\rho(-10\lambda_1 + 6\lambda_2),\\
\tfrac{\partial}{\partial u}(\lambda_2) &= a_4\rho(6\lambda_1 - 10\lambda_2),\\
\tfrac{\partial}{\partial v}(\lambda_2) &= -4a_5\rho\lambda_2.
\end{aligned}$$

where $\rho = \frac{(\lambda_1+\lambda_2)^{\frac{1}{2}}}{(\lambda_1-\lambda_2)^{\frac{7}{10}}}$ with $\lambda_1 + \lambda_2 \neq 0 \neq \lambda_1 - \lambda_2$. Put $X_1 = \frac{1}{\rho}\frac{\partial}{\partial u}$ and $X_2 = \frac{\partial}{\partial v}$ and define a connection on M as in Lemma 7. We also introduce h, $\nabla^\perp$ and shape operators by formulas described in Lemma 7. Then, there exist an immersion $\phi : D \to \mathbb{R}^4$ with planar ∇-geodesics. Conversely every definite immersion without affine umbilic or affine minimal points can be locally obtained in this way.

Proof: It is straightforward to check that all integrability conditions are satisfied. In orde to obtain the converse it is sufficient to check that

$$[\frac{(\lambda_1+\lambda_2)^{\frac{1}{2}}}{(\lambda_1-\lambda_2)^{\frac{7}{10}}}X_1, \frac{(\lambda_1+\lambda_2)^{\frac{1}{2}}}{(\lambda_1-\lambda_2)^{\frac{7}{10}}}X_2] = 0,$$

which can be done by a straightforward computation. ∎

4 Nondegenerate indefinite surfaces

In this section, we will assume that M^2 is a indefinite surface in $\mathbb{R}^4$. Then, we have the following lemma.

Lemma 9 *Let X_1, X_2, ξ_1, ξ_2 be a basis constructed as in Section 2. Then, $A_{h(v,v)}v$ is a multiple of v for every vector v if and only if*

$$\begin{aligned}
S_1X_1 &= \lambda_1 X_1,\\
S_1X_2 &= \mu X_1 + \lambda_1 X_2,\\
S_2X_1 &= \lambda_2 X_1 + \mu X_2,\\
S_2X_2 &= \lambda_2 X_2.
\end{aligned}$$

Proof: Since $\xi_1 = h(X_1, X_1)$ and $\xi_2 = -h(X_2, X_2)$, it follows immediately that we can write

$$\begin{aligned}
S_1X_1 &= \lambda_1 X_1,\\
S_2X_2 &= \lambda_2 X_2.
\end{aligned}$$

We now write

$$\begin{aligned}
S_1X_2 &= a_{11}X_1 + a_{12}X_2,\\
S_2X_1 &= a_{21}X_1 + a_{22}X_2,
\end{aligned}$$

and we take $v = aX_1 + bX_2$. Then it follows that

$$\begin{aligned}
A_{h(v,v)}v &= A_{a^2\xi_1+b^2\xi_2}(aX_1 + bX_2)\\
&= a^3\lambda_1X_1 + ba^2(a_{11}X_1 + a_{12}X_2) + ab^2(a_{21}X_1 + a_{22}X_2) + b^3\lambda_2X_2)\\
&= (a^3\lambda_1 + a^2ba_{11} + ab^2a_{21})X_1 + (b^3\lambda_2 + a^2ba_{12} + ab^2a_{22})X_2.
\end{aligned}$$

It follows that this is a multiple of the vector v if and only if

$$a^3 b\lambda_1 + a^2 b^2 a_{11} + ab^3 a_{21} - b^3 a\lambda_2 - a^3 b a_{12} - a^2 b^2 a_{22} = 0.$$

Since the above equation has to be valued for all values of a and b it follows that

$$\begin{aligned} a_{11} &= a_{22} = \mu, \\ a_{12} &= \lambda_1, \\ a_{21} &= \lambda_2. \end{aligned}$$

∎

Using the notation of the previous lemma, it follows also that

$$\begin{aligned} A_{h(v,v)}v &= (a^3\lambda_1 + a^2 b\mu + ab^2\lambda_2)X_1 + (b^3\lambda_2 + ab^2\mu + a^2 b\lambda_1)X_2 \\ &= (a^2\lambda_1 + ab\mu + b^2\lambda_2^2)v. \end{aligned}$$

This implies that the symmetric bilinear form $\tilde{g}$ introduced in the previous section is determined by

$$\begin{aligned} \tilde{g}(X_1, X_1) &= \lambda_1, \\ \tilde{g}(X_1, X_2) &= \mu, \\ \tilde{g}(X_2, X_2) &= \lambda_2. \end{aligned}$$

Since the conformal structure is now indefinite, there does not need to exists a basis X_1 and X_2 such that μ vanishes. Therefore, since we have the freedum to replace both X_1 and X_2 by arbitrary multiples, we have to consider the following cases:

(i) $\lambda_1 = \epsilon_1$ and $\lambda_2 = \epsilon_2$, where $\epsilon_1, \epsilon_2 = \pm 1$,

(ii) $\lambda_1 = \epsilon$, $\mu = 1$ and $\lambda_2 = 0$, where $\epsilon = \pm 1$,

(iii) $\lambda_1 = \epsilon$, $\mu = \lambda_2 = 0$ and $\Omega(X_1, X_2, \xi_1, \xi_2) = 1$,

(iv) $\lambda_1 = \lambda_2 = 0$ and $\Omega(X_1, X_2, \xi_1, \xi_2) = 1$. In this case we still have the freedom to replace $\{X_1, X_2, \xi_1, \xi_2\}$ by $\{rX_1, \frac{1}{r}X_2, r^2\xi_1, \frac{1}{r^2}\xi_2\}$. Here we consider still two cases. Either C vanishes identically, in which case the transversal plane is the equiaffine plane and hence by [5] we know that M is a product of two parabolas, or C does not vanish identically, in which case if necessary by interchanging X_1 and X_2, we can find a frame such that $\alpha(X_1) = -\frac{3}{4}$.

Depending on which type of frame we have locally, we call M an indefinite affine surface of Type 1,2,3 or 4. We now take the basis constructed before and introduce local functions by

$$\begin{aligned} \nabla_{X_1} X_1 &= a_1 X_1 + a_2 X_2, \\ \nabla_{X_1} X_2 &= a_3 X_1 + a_4 X_2, \\ \nabla_{X_1} X_2 &= a_5 X_1 + a_6 X_2, \\ \nabla_{X_2} X_2 &= a_7 X_1 + a_8 X_2, \\ \nabla^{\perp}_{X_1} \xi_1 &= d_1 \xi_1 + d_2 \xi_2, \\ \nabla^{\perp}_{X_1} \xi_2 &= d_3 \xi_1 + d_4 \xi_2, \\ \nabla^{\perp}_{X_2} \xi_1 &= d_5 \xi_1 + d_6 \xi_2, \\ \nabla^{\perp}_{X_2} \xi_2 &= d_7 \xi_1 + d_8 \xi_2. \end{aligned}$$

Except in Case 1 and 2, we have that $\Omega(X_1, X_2, \xi_1, \xi_2) = 1$, which implies that

$$a_1 + a_4 + d_1 + d_4 = 0 = a_5 + a_8 + d_5 + d_8 = 0.$$

Computing the Gauss, Codazzi and Ricci equations, together with the condition that C is divisible by h it then follows by a long but straightforward computation that

Lemma 10 *Let $\phi : (M^2, \nabla) \to \mathbb{R}^4$ be an indefinite affine surface whose geodesics are planar curves in $\mathbb{R}^4$ which is of Type 1. Then we have that*

$$\begin{array}{llllll}
a_2 = 0, & a_3 = \frac{2}{3}a_8, & a_4 = \frac{1}{3}(a_1 - \epsilon_2\mu a_8), & a_5 = \frac{1}{3}(a_8 - \epsilon_1\mu a_1) & a_6 = \frac{2}{3}a_1, & a_7 = 0, \\
d_1 = 0, & d_2 = 0, & d_3 = 0, & & d_4 = -\frac{2}{3}\epsilon_2\mu a_8, & \\
d_5 = -\frac{2}{3}\epsilon_1 a_1\mu, & d_6 = 0, & d_7 = 0, & & d_8 = 0, &
\end{array}$$

where the functions a_1, a_8 and μ satisfy the following system of differential equations:

$$\begin{aligned}
X_1(a_1) &= -\tfrac{3}{2}\epsilon_1 + \tfrac{1}{3}a_1^2, \\
X_2(a_1) &= \tfrac{9}{4}\mu + \tfrac{1}{3}a_1(a_8 - \epsilon_1\mu a_1), \\
X_1(a_8) &= \tfrac{9}{4}\mu + \tfrac{1}{3}a_8(a_1 - \epsilon_2\mu a_8), \\
X_2(a_8) &= -\tfrac{3}{2}\epsilon_2 + \tfrac{1}{3}a_8^2, \\
X_1(\mu) &= -\tfrac{1}{3}\mu(2a_1 + \epsilon_2\mu a_8), \\
X_2(\mu) &= -\tfrac{1}{3}\mu(2a_8 + \epsilon_1\mu a_1).
\end{aligned}$$

Theorem 5 *There does not exist a indefinite affine immersion $\phi : (M^2, \nabla) \to \mathbb{R}^4$ whose geodesics are planar curves in $\mathbb{R}^4$ and which is of Type 2.*

Proof: First, we deduce from the fact that C is divisible by h that

$$\begin{aligned}
d_2 &= a_2 = d_3 = d_6 = d_7 = a_7 = 0, \\
d_1 &= 2a_1 + 3\alpha_1, \\
d_8 &= 2a_8 + 3\alpha_2, \\
a_3 &= -\alpha_2, \\
d_5 &= 2a_5 + \alpha_2, \\
d_4 &= 2a_4 + \alpha_1, \\
a_6 &= -\alpha_1,
\end{aligned}$$

where $\alpha_1 = \alpha(X_1)$ and $\alpha_2 = \alpha(X_2)$. Next, we look at the euqations of Codazzi for the shape operators. These imply that

$$\begin{aligned}
\alpha_1 &= -\tfrac{2}{3}a_1, \\
\alpha_2 &= -\tfrac{1}{4}(a_5 + a_8), \\
a_4 &= \tfrac{5}{3}a_1 + \tfrac{7}{4}\epsilon a_5 - \tfrac{1}{4}\epsilon a_8.
\end{aligned}$$

Next, we compute the equations of Gauss and Ricci. Solving for the derivatives of the unknown remaining functions it follows that

$$X_1(a_1) = -\tfrac{3}{2}\epsilon + \tfrac{1}{3}a_1^2, \tag{60}$$
$$X_2(a_1) = \tfrac{9}{4} + a_1 a_5, \tag{61}$$
$$X_1(a_5) = \tfrac{1}{12}(15 + 13a_1a_5 + 21\epsilon a_5^2 + a_1a_8 - 3\epsilon a_5 a_8), \tag{62}$$
$$X_2(a_5) = \tfrac{1}{8}(5a_5^2 - 2a_5a_8 + a_8^2), \tag{63}$$
$$X_1(a_8) = \tfrac{1}{12}(57 + 7a_1a_5 + 19a_1a_8 + 21\epsilon a_5 a_8 - 3\epsilon a_8^2), \tag{64}$$
$$X_2(a_8) = \tfrac{1}{8}(-7a_5^2 + 6a_5a_8 + 5a_8^2). \tag{65}$$

Since

$$X_1(X_2(a_5)) - X_2(X_1(a_5)) = (\nabla_{X_1}X_2 - \nabla_{X_2}X_1)a_5,$$

it follows that

$$a_5 = \tfrac{3}{21}a_8 - \tfrac{8}{21}\epsilon a_1.$$

Checking the compatibility of the above equation with (62) and (63), it follows that a_1 can not vanish and that

$$a_8 = \frac{-189+8\epsilon a_1^2}{24a_1}.$$

Checking the compatibility of the above expression with (64) and (65) a contradiction now follows. ∎

Lemma 11 *Let $\phi : (M^2, \nabla) \to \mathbb{R}^4$ be a indefinite immersion whose geodesics are planar curves in $\mathbb{R}^4$ which is of Type 3. Then we have that*

$$\begin{array}{llll} a_2 = 0, & a_3 = \frac{6}{5}a_8, & a_4 = -\frac{1}{9}a_1, & \\ a_5 = \frac{3}{5}a_8, & a_6 = \frac{2}{3}a_1, & a_7 = 0, & \\ d_1 = 0, & d_2 = 0, & d_3 = 0, & d_4 = -\frac{8}{9}a_1, \\ d_5 = 0, & d_6 = 0, & d_7 = 0, & d_8 = -\frac{8}{5}a_8, \end{array}$$

where the functions a_1 and a_8 satisfy the following system of differential equations:

$$\begin{aligned} X_1(a_1) &= -\tfrac{3}{2}\epsilon + \tfrac{1}{3}a_1^2, \\ X_2(a_1) &= \tfrac{3}{5}a_1 a_8, \\ X_1(a_8) &= -\tfrac{1}{9}a_1 a_8, \\ X_2(a_8) &= -\tfrac{1}{5}a_8^2. \end{aligned}$$

Lemma 12 *Let $\phi : (M^2, \nabla) \to \mathbb{R}^4$ be a indefinite immersion whose geodesics are planar curves in $\mathbb{R}^4$ and which is of Type 4. Then either M is a product of two parabolas or we have that $\alpha(X_1) = -3/4$, $\mu = 0$ and that*

$$\begin{array}{llll} a_1 = \frac{3}{4}, & a_2 = 0, & a_3 = \frac{3}{4}a_8, & a_4 = \frac{1}{4}, \\ a_5 = 0, & a_6 = \frac{3}{4}, & a_7 = 0, & \\ d_1 = -\frac{3}{4}, & d_2 = 0, & d_3 = 0, & d_4 = -\frac{1}{4}, \\ d_5 = -\frac{3}{4}a_8, & d_6 = 0, & d_7 = 0, & d_8 = -\frac{1}{4}a_8, \end{array}$$

where the function a_8 satisfies the following system of differential equations:

$$\begin{aligned} X_1(a_8) &= \tfrac{1}{4}a_8, \\ X_2(a_8) &= \tfrac{1}{4}a_8^2. \end{aligned}$$

Theorem 6 *Consider $D \subset \mathbb{R}^2$ and let $a_4, a_5, \lambda_1, \lambda_2$ be a solution of*

$$\begin{aligned} \tfrac{\partial}{\partial u}(a_1) &= \rho(-\tfrac{3}{2}\epsilon_1 + \tfrac{1}{3}a_1^2), \\ \tfrac{\partial}{\partial v}(a_1) &= \rho(\tfrac{9}{4}\mu + \tfrac{1}{3}a_1(a_8 - \epsilon_1\mu a_1)), \\ \tfrac{\partial}{\partial u}(a_8) &= \rho(\tfrac{9}{4}\mu + \tfrac{1}{3}a_8(a_1 - \epsilon_2\mu a_8)), \\ \tfrac{\partial}{\partial v}(a_8) &= \rho(-\tfrac{3}{2}\epsilon_2 + \tfrac{1}{3}a_8^2), \\ \tfrac{\partial}{\partial u}(\mu) &= -\tfrac{1}{3}\rho\mu(2a_1 + \epsilon_2\mu a_8), \\ \tfrac{\partial}{\partial v}(\mu) &= -\tfrac{1}{3}\rho\mu(2a_8 + \epsilon_1\mu a_1), \\ \tfrac{\partial}{\partial u}(\rho) &= \tfrac{1}{3}\rho^2(a_1 + \epsilon_2\mu a_8), \\ \tfrac{\partial}{\partial v}(\rho) &= \tfrac{1}{3}\rho^2(a_8 + \epsilon_1\mu a_1). \end{aligned}$$

Put $X_1 = \frac{1}{\rho}\frac{\partial}{\partial u}$ and $X_2 = \frac{\partial}{\partial v}$ and define a connection on M as in Lemma 10. We also introduce h, $\nabla^\perp$ and shape operators by formulas described in Lemma 10. Then, there exist an immersion $\phi : D \to \mathbb{R}^4$ with planar ∇-geodesics. Conversely every indefinite immersion which is of Type 1 can be locally obtained in this way.

Proof: It is straightforward to check that all integrability conditions as well as the Gauss, Codazzi and Ricci equation are satisfied. Therefore applying the standard existence and uniqueness theorem completes the first part of the proof. In orde to obtain the converse we first introduce a function ρ by

$$X_1(\rho) = \tfrac{1}{3}\rho(a_1 + \epsilon_2\mu a_8)$$
$$X_2(\rho) = \tfrac{1}{3}\rho(a_8 + \epsilon_1\mu a_1)$$

A function ρ satisfying the above system of differential equations exists if and only if

$$X_1(X_2(\log(\rho))) - X_2(X_1(\log(\rho))) = \tfrac{1}{3}(a_8 + \epsilon_1\mu a_1)X_1(\log\rho) - \tfrac{1}{3}(a_1 + \epsilon_2\mu a_8)X_2(\log\rho).$$

Since the righthandside of the above equations vanishes, such a function ρ exists if and only if

$$\begin{aligned}
0 &= X_1(X_2(\log\rho)) - X_2(X_1(\log\rho)) \\
&= \tfrac{1}{3}(\tfrac{9}{4}\mu + \tfrac{1}{3}a_8(a_1 - \epsilon_2\mu a_8)) - \tfrac{1}{9}\epsilon_1 a_1\mu(2a_1 + \epsilon_2\mu a_8) + \tfrac{1}{3}\epsilon_1\mu(-\tfrac{3}{2}\epsilon_1 + \tfrac{1}{3}a_1^2) \\
&\quad - \tfrac{1}{3}(\tfrac{9}{4}\mu + \tfrac{1}{3}a_1(a_8 - \epsilon_1\mu a_1)) + \tfrac{1}{9}\epsilon_2 a_8\mu(2a_8 + \epsilon_1\mu a_1) - \tfrac{1}{3}\epsilon_2\mu(-\tfrac{3}{2}\epsilon_2 + \tfrac{1}{3}a_8^2).
\end{aligned}$$

Since

$$\begin{aligned}
[\rho X_1, \rho X_2] &= \rho^2[X_1, X_2] + \rho X_1(\rho)X_2 - \rho X_2(\rho)X_1 \\
&= \rho^2(\tfrac{1}{3}(a_8 + \epsilon_1\mu a_1)X_1 - \tfrac{1}{3}(a_1 + \epsilon_2\mu a_8)X_2) + \rho X_1(\rho)X_2 - \rho X_2(\rho)X_1 \\
&= 0,
\end{aligned}$$

it follows that we can introduce coordinates u and v such that $\frac{\partial}{\partial u} = \rho X_1$ and $\frac{\partial}{\partial v} = \rho X_2$. From this the converse follows immediately. ∎

Theorem 7 *Let $\phi : (M^2, \nabla) \to \mathbb{R}^4$ be a indefinite immersion whose geodesics are planar curves in $\mathbb{R}^4$ which is of Type 3. Then, M is locally affine congruent to either*

(i) the surface $\frac{1}{(\frac{1}{2}\epsilon u^2 - \frac{6}{5}v)}(\frac{1}{2}u^2, \frac{1}{2}v^2, u, 1)$,

(ii) the surface $e^{\sqrt{2}u}(\frac{1}{2}v^2, v, 1, e^{-\frac{1}{\sqrt{2}}u})$,

(iii) the surface $(\cosh(\frac{1}{\sqrt{2}}u)^{-2}(\frac{1}{2}v^2, v, 1, (\sinh(\frac{1}{\sqrt{2}}u)))$,

(iv) the surface $(\sinh(\frac{1}{\sqrt{2}}u)^{-2}(\frac{1}{2}v^2, v, 1, (\cosh(\frac{1}{\sqrt{2}}u)))$,

(v) the surface $(\cos(\frac{1}{\sqrt{2}}u)^{-2}(\frac{1}{2}v^2, v, 1, (\sin(\frac{1}{\sqrt{2}}u)))$.

Proof: We again consider two cases. First, we assume that $a_8 \neq 0$ in a neighborhood of the point p. Then, it follows that

$$[a_8^{-3}X_1, a_8^{-7}X_2] = a_8^{-11}(\tfrac{3}{5}a_8X_1 - \tfrac{7}{9}a_1X_2) + \tfrac{7}{9}a_8^{-11}a_1X_2 - \tfrac{3}{5}a_8^{-12}X_1 = 0. \tag{66}$$

Therefore there exist coordinates u and v such that $\frac{\partial}{\partial u} = a_8^{-3}X_1$ and $\frac{\partial}{\partial v} = a_8^{-7}X_2$. Then, the functions a_1 and a_8 are determined by

$$\frac{\partial a_1}{\partial u} = a_8^{-3}(-\tfrac{3}{2}\epsilon + \tfrac{1}{3}a_1^2), \tag{67}$$
$$\frac{\partial a_1}{\partial v} = \tfrac{3}{5}a_1a_8^{-6}, \tag{68}$$
$$\frac{\partial a_8}{\partial u} = -\tfrac{1}{9}a_1a_8^{-2}, \tag{69}$$
$$\frac{\partial a_8}{\partial v} = -\tfrac{1}{5}a_8^{-5}. \tag{70}$$

From (67) upto (70) it then follows that

$$\frac{\partial}{\partial u}(a_8^3 a_1) = -\tfrac{3}{2}\epsilon,$$
$$\frac{\partial}{\partial v}(a_8^3 a_1) = 0.$$

Hence by applying a translation of the u coordinate we may assume that $a_8^3 a_1 = -\frac{3}{2}\epsilon u$. The equations (67) up to (70) then reduce to the following system of differential equations for a_8:

$$\frac{\partial}{\partial u} a_8 = \tfrac{1}{6}\epsilon u a_8^{-5},$$
$$\frac{\partial}{\partial v} a_8 = -\tfrac{1}{5} a_8^{-5}.$$

Solving the above yields that up to a translation of the v-coordinate we have that

$$a_8^6 = \tfrac{1}{2}\epsilon u^2 - \tfrac{6}{5}v.$$

It then follows that the immersion ϕ is determined by the following system of differential equations:

$$\phi_{uu} = \tfrac{4}{3} a_1 a_8^{-3}\phi_u + a_8^{-6}\xi_1 \tag{71}$$
$$\phi_{uv} = \tfrac{6}{5} a_8^{-6}\phi_u + \tfrac{2}{3} a_1 a_8^{-3}\phi_v \tag{72}$$
$$\phi_{vv} = \tfrac{12}{5} a_8^{-6}\phi_v + a_8^{-14}\xi_2 \tag{73}$$
$$\xi_{1u} = -\epsilon\phi_u \tag{74}$$
$$\xi_{2u} = -\tfrac{8}{9} a_1 a_8^{-3}\xi_2 \tag{75}$$
$$\xi_{1v} = -\epsilon\phi_v \tag{76}$$
$$\xi_{2v} = -\tfrac{8}{5} a_8^{-6}\xi_2 \tag{77}$$

Solving first (74), (75), (76) and (77) it follows that there exist constant vectors A_1 and A_2 such that $\xi_1 + \epsilon\phi = A_1$ and $a_8^{-8}\xi_2 = A_2$. Hence the system of differential equations reduces to

$$(a_8^6\phi)_{uu} = -2\epsilon u\phi_u - \epsilon\phi + A_1 + \epsilon\phi + 2u\epsilon\phi_u = A_1,$$
$$(a_8^6\phi)_{uv} = \tfrac{6}{5}\phi_u - \epsilon u\phi_v + u\epsilon\phi_v - \tfrac{6}{5}\phi_u = 0,$$
$$(a_8^6\phi)_{vv} = A_2.$$

Integrating the above expression, we get that ϕ is congruent with

$$(\tfrac{1}{2}\epsilon u^2 - \tfrac{6}{5}v)\phi = \tfrac{1}{2}u^2 A_1 + \tfrac{1}{2}v^2 A_2 + A_3 u + A_4 v + A_5.$$

Hence by applying an affine transformation ϕ is congruent with

$$\frac{1}{(\frac{1}{2}\epsilon u^2 - \frac{6}{5}v)}(\tfrac{1}{2}u^2, \tfrac{1}{2}v^2, u, 1).$$

Next, we consider the case that $a_8 = 0$ on a neighborhood of the point p. In that case, we have that

$$[X_1, X_2] = -\tfrac{7}{9}a_1 X_1,$$

and we see that the integrability conditions for

$$X_1(\rho) = \tfrac{7}{9}a_1\rho,$$
$$X_2(\rho) = 0,$$

are satisfied. Consequently there exist coordinates u and v such that $\frac{\partial}{\partial u} = X_1$ and $\frac{\partial}{\partial v} = \rho X_2$. It also follows from the above that ρ and a_1 only depend on the variable u and are determined as solutions of

$$\frac{\partial\rho}{\partial u} = \tfrac{7}{9}a_1\rho,$$
$$\frac{\partial a_1}{\partial u} = -\tfrac{3}{2}\epsilon + \tfrac{1}{3}a_1^2.$$

Remark that if necessary by replacing X_1 with $-X_1$, we may assume that a_1 is nonnegative. Therefore, by choosing suitable initial conditions for the function ρ and a translation of the u-coordinate, it follows that either

(i) $a_1 = \frac{3}{\sqrt{2}}$ and $\rho = e^{\frac{7}{3\sqrt{2}}u}$,

(ii) $a_1 = -\frac{3}{\sqrt{2}}\frac{\sinh(\frac{1}{\sqrt{2}}u)}{\cosh(\frac{1}{\sqrt{2}}u)}$ and $\rho = (\cosh(\frac{1}{\sqrt{2}}u))^{-\frac{7}{3}}$,

(iii) $a_1 = -\frac{3}{\sqrt{2}}\frac{\cosh(\frac{1}{\sqrt{2}}u)}{\sinh(\frac{1}{\sqrt{2}}u)}$ and $\rho = (\sinh(\frac{1}{\sqrt{2}}u))^{-\frac{7}{3}}$,

(iv) $a_1 = \frac{3}{\sqrt{2}}\frac{\sin(\frac{1}{\sqrt{2}}u)}{\cos(\frac{1}{\sqrt{2}}u)}$ and $\rho = (\cos(\frac{1}{\sqrt{2}}u))^{-\frac{7}{3}}$,

where in the first 3 Cases we have $\epsilon = 1$ and in the last case, we have that $\epsilon = -1$. We then see that in all four cases the immersion ϕ is determined by the following system of differential equations:

$$\phi_{uu} = a_1\phi_u + \xi_1, \tag{78}$$
$$\phi_{uv} = \tfrac{2}{3}a_1\phi_v, \tag{79}$$
$$\phi_{vv} = \rho^2\xi_2, \tag{80}$$
$$\xi_{1u} = -\epsilon\phi_u, \tag{81}$$
$$\xi_{2u} = -\tfrac{8}{9}a_1\xi_2, \tag{82}$$
$$\xi_{1v} = -\epsilon\phi_v, \tag{83}$$
$$\xi_{2v} = 0. \tag{84}$$

Solving first (81), (82), (83) and (84) it follows that there exist constant vectors A_1 and A_2 such that $\xi_1 + \epsilon\phi = A_1$ and $\rho^{\frac{8}{7}}\xi_2 = A_2$. Hence the system of differential equations reduces to

$$\phi_{uu} = a_1\phi_u - \epsilon\phi + A_1,$$
$$(\rho^{-\frac{6}{7}}\phi)_{uv} = 0,$$
$$(\rho^{-\frac{6}{7}}\phi)_{vv} = A_2.$$

Replacing ϕ by $\phi - \epsilon A_1$ in the above system we deduce that by a translation we may assume that $A_1 = 0$. Solving the above system of differential equations, it follows that

$$\phi = \tfrac{1}{2}A_2v^2\rho^{\frac{6}{7}} + A_3v\rho^{\frac{6}{7}} + \rho^{\frac{6}{7}}C(u),$$

where C is a solution of

$$C''(u) + \tfrac{1}{3}a_1C'(u) = 0.$$

Thus

$$C'(u) = A_4\rho^{-\frac{3}{7}}.$$

Looking at the different cases, we obtain either

$$D'(u) = A_4e^{-\frac{1}{\sqrt{2}}u},$$
$$D'(u) = A_4(\cosh(\tfrac{1}{\sqrt{2}}u),$$
$$D'(u) = A_4(\sinh(\tfrac{1}{\sqrt{2}}u),$$
$$D'(u) = A_4(\cos(\tfrac{1}{\sqrt{2}}u).$$

Thus M is affine equivalent with either

$$e^{\sqrt{2}u}(\tfrac{1}{2}v^2, v, 1, e^{-\frac{1}{\sqrt{2}}u}),$$
$$(\cosh(\tfrac{1}{\sqrt{2}}u)^{-2}(\tfrac{1}{2}v^2, v, 1, (\sinh(\tfrac{1}{\sqrt{2}}u))),$$
$$(\sinh(\tfrac{1}{\sqrt{2}}u)^{-2}(\tfrac{1}{2}v^2, v, 1, (\cosh(\tfrac{1}{\sqrt{2}}u))),$$
$$(\cos(\tfrac{1}{\sqrt{2}}u)^{-2}(\tfrac{1}{2}v^2, v, 1, (\sin(\tfrac{1}{\sqrt{2}}u)).)$$

∎

Theorem 8 *Let $\phi : (M^2, \nabla) \to \mathbb{R}^4$ be a indefinite immersion whose geodesics are planar curves in $\mathbb{R}^4$ and which is of Type 4. Then M is locally affine congruent to either*

(i) a product of two parabolas,

(ii) the surface $(-\frac{2}{3}\frac{u^2}{u+v}, -\frac{2}{3}\frac{v^2}{u+v}, \frac{u}{u+v}, 1)$,

(iii) the surface $\phi(u,v) = (e^{\frac{3}{4}u}\frac{v^2}{2}, e^{\frac{3}{4}u}v, e^{-\frac{3}{4}u}, e^{\frac{3}{4}u})$.

Proof: We have that $[X_1, X_2] = \frac{3}{4}a_8X_1 - \frac{1}{2}X_2$. First we consider the case that $a_8 \neq 0$ in a neighborhood of the point p. We have that

$$\begin{aligned}[a_8^3X_1, a_8^2X_2] &= a_8^5(\tfrac{3}{4}a_8X_1 - \tfrac{1}{2}X_2) + 2a_8^4(\tfrac{1}{4}a_8)X_2 - 3a_8^4(\tfrac{1}{4}a_8^2)X_1\\ &= 0.\end{aligned}$$

Therefore, there exist coordinates u and v such that $\frac{\partial}{\partial u} = a_8^3X_1$ and $\frac{\partial}{\partial v} = a_8^2X_2$. The function a_8 is then determined by

$$\begin{aligned}\tfrac{\partial a_8}{\partial u} &= \tfrac{1}{4}a_8^4,\\ \tfrac{\partial a_8}{\partial v} &= \tfrac{1}{4}a_8^4.\end{aligned}$$

Solving the above system, we find after a translation in the u or v coordinate that

$$-\tfrac{1}{3}a_8^{-3} = \tfrac{1}{4}(u+v).$$

The immersion ϕ is then determined by

$$\begin{aligned}
\phi_{uu} &= \tfrac{3}{2}a_8^3\phi_u + a_8^6\xi_1, && (85)\\
\phi_{uv} &= \tfrac{3}{4}a_8^3\phi_u + \tfrac{3}{4}a_8^3\phi_v, && (86)\\
\phi_{vv} &= \tfrac{3}{2}a_8^3\phi_v + a_8^4\xi_2, && (87)\\
\xi_{1u} &= -\tfrac{3}{4}a_8^3\xi_1, && (88)\\
\xi_{2u} &= -\tfrac{1}{4}a_8^3\xi_2, && (89)\\
\xi_{1v} &= -\tfrac{3}{4}a_8^3\xi_1, && (90)\\
\xi_{2v} &= -\tfrac{1}{4}a_8^3\xi_2. && (91)
\end{aligned}$$

Solving first (88), (89), (90) and (91) it follows that there exist constant vectors A_1 and A_2 such that $a_8^3\xi_1 = A_1$ and $a_8\xi_2 = A_2$. Hence the system of differential equations reduces to

$$\begin{aligned}(a_8^{-3}\phi)_{uu} &= A_1,\\ (a_8^{-3}\phi)_{uv} &= 0,\\ (a_8^{-3}\phi)_{vv} &= A_2.\end{aligned}$$

Hence integrating yields

$$-\tfrac{3}{4}(u+v)\phi = \tfrac{1}{2}A_1u^2 + \tfrac{1}{2}A_2v^2 + A_3u + A_4v + A_5.$$

Hence by an affine transformation and translation, we get that ϕ is congruent with

$$\phi(u,v) = (-\tfrac{2}{3}\tfrac{u^2}{u+v}, -\tfrac{2}{3}\tfrac{v^2}{u+v}, \tfrac{u}{u+v}, 1).$$

Finally, we consider the case that $a_8 = 0$ on a neighborhood of the point p. In that case $[X_1, X_2] = -\frac{1}{2}X_2$ and it is straightforward to check that there exists a function ρ such that $X_1(\rho) = \frac{1}{2}\rho$ and $X_2(\rho) = 0$. Then, we have that

$$[X_1, \rho X_2] = -\tfrac{1}{2}\rho X_2 + \tfrac{1}{2}\rho X_2 = 0,$$

and hence there exist coordinates u and v such that $\frac{\partial}{\partial u} = X_1$ and $\frac{\partial}{\partial v} = \rho X_2$. The function ρ is then determined by

$$\begin{aligned}\tfrac{\partial \rho}{\partial u} &= \tfrac{1}{2}\rho,\\ \tfrac{\partial \rho}{\partial v} &= 0.\end{aligned}$$

Solving the above system, we find that we can take $\rho = e^{\frac{1}{2}u}$. The immersion ϕ is then determined by

$$\phi_{uu} = \tfrac{3}{4}\phi_u + \xi_1, \tag{92}$$
$$\phi_{uv} = \tfrac{3}{4}\phi_v, \tag{93}$$
$$\phi_{vv} = e^u \xi_2, \tag{94}$$
$$\xi_{1u} = -\tfrac{3}{4}\xi_1, \tag{95}$$
$$\xi_{2u} = -\tfrac{1}{4}\xi_2, \tag{96}$$
$$\xi_{1v} = 0, \tag{97}$$
$$\xi_{2v} = 0. \tag{98}$$

Solving first (95), (96), (97) and (98) it follows that there exist constant vectors A_1 and A_2 such that $e^{\frac{3}{4}u}\xi_1 = A_1$ and $e^{\frac{1}{4}u}\xi_2 = A_2$. Thus

$$\phi(u, v) = e^{\frac{3}{4}u} A_2 \tfrac{v^2}{2} + e^{\frac{3}{4}u} A_3 v + C_1(u),$$

where C_1 satisfies

$$C_1''(u) = \tfrac{3}{4}C_1'(u) + A_1 e^{-\frac{3}{4}u}.$$

Therefore after applying a translation and an affine transformation, we see that

$$\phi(u, v) = (e^{\frac{3}{4}u}\tfrac{v^2}{2}, e^{\frac{3}{4}u}v, e^{-\frac{3}{4}u}, e^{\frac{3}{4}u}).$$

∎

5 Appendix

In the previous sections, two classes were described only by applying an existence and uniqueness theorem. Remark that if necessary by complexifying everything, the positive definite case can be treated similarly to the indefinite case. For that reason, we will restrict ourselves here to the indefinite case and more in particular to the class of examples described in Theorem 6. First, we show how the system of differential equations for the functions ρ, μ, a_1 and a_8 can be solved. For this purpose, we introduce two new functions f and g by

$$\begin{aligned} f &= \mu^2 \rho^2,\\ g &= \mu \rho^2.\end{aligned}$$

It then follows that

$$\tfrac{\partial f}{\partial u} = -\tfrac{2}{3}\mu^2\rho^3 a_1, \tag{99}$$
$$\tfrac{\partial f}{\partial v} = -\tfrac{2}{3}\mu^2\rho^3 a_8, \tag{100}$$
$$\tfrac{\partial g}{\partial u} = \tfrac{1}{3}\mu^2\rho^3 \epsilon_2 a_8, \tag{101}$$
$$\tfrac{\partial g}{\partial v} = \tfrac{1}{3}\mu^2\rho^3 \epsilon_1 a_1. \tag{102}$$

Consequently, we have that

$$\frac{\partial f}{\partial u} = -2\epsilon_1 \frac{\partial g}{\partial v},$$
$$\frac{\partial f}{\partial v} = -2\epsilon_2 \frac{\partial g}{\partial u}.$$

Consequently $\epsilon_1 \frac{\partial^2 f}{\partial u^2} = \epsilon_2 \frac{\partial^2 f}{\partial v^2}$.

Now we consider two cases. First, we assume that $\epsilon_1 = \epsilon_2 = \epsilon$. In that case it follows from the above equation that we can write

$$f(u,v) = \alpha_1(u+v) + \alpha_2(u-v),$$
$$g(u,v) = -\tfrac{1}{2}\epsilon(\alpha_1(u+v) - \alpha_2(u-v)).$$

We then see from (99) and (100) that

$$\mu\rho a_1 = 3\epsilon \frac{\alpha_1'(u+v)+\alpha_2'(u-v)}{\alpha_1(u+v)-\alpha_2(u-v)}$$
$$\mu\rho a_8 = 3\epsilon \frac{\alpha_1'(u+v)-\alpha_2'(u-v)}{\alpha_1(u+v)-\alpha_2(u-v)}$$

Computing now the derivatives with respect to u and v of the above expressions and expressing everything again in terms of α_1 and α_2, it follows that as functions of 1 variable t the functions α_1 and α_2 are determined by

$$(\alpha_1'(t))^2 = \tfrac{1}{2}\epsilon\alpha_1(t)^3 + C_1\alpha_1 + C_2$$
$$(\alpha_2'(t))^2 = \tfrac{1}{2}\epsilon\alpha_2(t)^3 + C_1\alpha_2 + C_2$$

Both of the integrals can be solved in terms of elliptic functions.

In the case that $\epsilon_1 \neq \epsilon_2$, if necessary by interchanging X_1 and X_2, we may assume that $\epsilon_1 = 1$ and $\epsilon_2 = -1$. We then proceed as follows. The differential equations then imply that

$$\frac{\partial}{\partial u} f = \frac{\partial}{\partial v}(-2g),$$
$$\frac{\partial}{\partial v} f = -\frac{\partial}{\partial u}(-2g).$$

Consequently the function $f - 2ig$ satisfies the Cauchy-Riemann equations. Hence $\phi = f - 2ig$ is a holomorphic function in the variable $z = u + iv$. It then follows straightforwardly that

$$\mu\rho a_1 = \frac{3i}{\phi - \bar{\phi}}\left(\frac{\partial \phi}{\partial z} + \frac{\partial \bar{\phi}}{\partial \bar{z}}\right).$$

Deriving now the above equation with respect to u and v we deduce that ϕ is a solution of

$$\left(\frac{\partial \phi}{\partial z}\right)^2 = -\tfrac{1}{4}\phi^3 + \tfrac{1}{2}C_1\phi + C_2,$$

where C_1 and C_2 are real numbers, which can be solved in terms of elliptic functions.

Now we look at the system of equations determining the immersion ϕ. It follows immediately that

$$(\epsilon_1\phi + \xi_1)_u = 0, \tag{103}$$
$$(\epsilon_2\phi + \xi_2)_v = 0, \tag{104}$$
$$(\epsilon_1\phi + \xi_1)_v = -\mu\phi_u - \tfrac{2}{3}a_1\rho\mu\epsilon_1\xi_1, \tag{105}$$
$$(\epsilon_2\phi + \xi_2)_u = -\mu\phi_v - \tfrac{2}{3}a_8\rho\mu\epsilon_2\xi_2. \tag{106}$$

It now follows from (103) and (104) that there exist curves β_1, depending only on v, and β_2, depending only on u such that

$$\beta_1(v) = \epsilon_1\phi + \xi_1$$
$$\beta_2(u) = \epsilon_2\phi + \xi_2.$$

Introducing now $\psi = f\phi$ it follows from (105) and (106) that

$$\psi_u = -\tfrac{f}{\mu}\beta_1'(v) - \tfrac{2}{3}\epsilon_1 a_1 \rho f \beta_1(v), \tag{107}$$
$$\psi_v = -\tfrac{f}{\mu}\beta_2'(u) - \tfrac{2}{3}\epsilon_2 a_8 \rho f \beta_2(u). \tag{108}$$

Computing hten ψ_{uv}, on the one hand using the above equations, and on the other hand using that $\phi_{uv} = \rho(\frac{2}{3}a_8\phi_u + \frac{2}{3}a_1\phi_v)$, it follows that

$$f\phi = \psi = \tfrac{2}{3}\beta_1''(v) + \tfrac{2}{3}a_1\epsilon_1\mu\rho\beta_1'(v) + \epsilon_1\mu^2\rho^2\beta_1(v),$$
$$f\phi = \psi = \tfrac{2}{3}\beta_2''(u) + \tfrac{2}{3}a_8\epsilon_2\mu\rho\beta_2'(u) + \epsilon_2\mu^2\rho^2\beta_2(u).$$

References

[1] K. Nomizu and U. Pinkall. On the geometry of affine immersions. *Math. Z.*, 195:165–178, 1987.

[2] K. Nomizu and U. Pinkall. Cubic form theorem for affine immersions. *Result. Math.*, 13:338–362, 1988.

[3] K. Nomizu and L. Vrancken. A new equiaffine theory for surfaces in $\mathbb{R}^4$. *International J. Math.*, 4:127–165, 1993.

[4] C. Scharlach and L. Vrancken. Centroaffine surfaces in $\mathbb{R}^4$ with planar ∇-geodesics. *Proc. Amer. Math. Soc.*, 126:213–219, 1998.

[5] L. Vrancken. Affine surfaces whose geodesics are planar curves. *Proc. Amer. Math. Soc.*, 123:3851–3854, 1995.

[6] K. Weise. Der Beruehrungstensor zweier Flaechen und die Affingeometrie der F_p im A_n. I. *Math. Z.*, 43:469–480, 1937.

[7] K. Weise. Der Beruehrungstensor zweier Flaechen und die Affingeometrie der F_p im A_n. II. *Math. Z.*, 44:161–184, 1938.